

# THE STRONG REFLECTING PROPERTY AND HARRINGTON'S PRINCIPLE

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**ABSTRACT.** In this paper we characterize the strong reflecting property for  $L$ -cardinals for all  $\omega_n$ , characterize Harrington's Principle  $HP(L)$  and its generalization and discuss the relationship between the strong reflecting property for  $L$ -cardinals and Harrington's Principle  $HP(L)$ .

## 1. INTRODUCTION AND PRELIMINARIES

The notion of the strong reflecting property for  $L$ -cardinals is introduced in [1, Definition 2.8]. The motivation of introducing this notion is to force a set model of Harrington's Principle,  $HP(L)$  for short (cf. Definition 3.1), over higher order arithmetic (cf. Definition 1.1). However the proof of The Main Theorem in [1] uses very little knowledge about the strong reflecting property for  $L$ -cardinals. In this paper, in Section 2 we develop the full theory of the strong reflecting property for  $L$ -cardinals and characterize  $SRP^L(\omega_n)$  for  $n \in \omega$  (see Proposition 2.8, Proposition 2.11, Theorem 2.17 and Theorem 2.23). We also generalize some results on  $SRP^L(\gamma)$  to  $SRP^M(\gamma)$  for other inner models  $M$  (see Theorem 2.20 and Theorem 2.27).

In Section 3, we define the generalized Harrington's Principle  $HP(M)$  for any inner model  $M$ , give characterizations of  $HP(M)$  for some well known inner models (see Theorem 3.3 and 3.9) and show that, in some cases, this generalized principle fails (see Corollary 3.11 and Theorem 3.14). In Section 4, we discuss the relationship between the strong reflecting property for  $L$ -cardinals and Harrington's Principle  $HP(L)$ .

Our definitions and notations are standard. We refer to textbooks such as [8], [10] and [11] for the definitions and notations we use. For the definition of admissible set and admissible ordinal, see [4]. For notions of large cardinals, see [10]. Our notations about forcing are standard (cf. [8] and [3]). For the theory of  $0^\sharp$  see [4] and [8]. Recall that  $0^\sharp$  is the unique well founded remarkable  $E.M.$  set, and  $0^\sharp$  exists if and only if for some uncountable limit ordinal  $\lambda$ ,  $L_\lambda$  has an uncountable set of indiscernibles (cf. [4] and [8]). For the theory of  $0^\dagger$  see [10].

**Definition 1.1.** ([1])

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- (i)  $Z_2 = ZFC^- + \text{Any set is Countable.}^1$
- (ii)  $Z_3 = ZFC^- + \mathcal{P}(\omega) \text{ exists} + \text{Any set is of cardinality } \leq \beth_1.$
- (iii)  $Z_4 = ZFC^- + \mathcal{P}(\mathcal{P}(\omega)) \text{ exists} + \text{Any set is of cardinality } \leq \beth_2.$

$Z_2, Z_3$  and  $Z_4$  are the corresponding axiomatic systems for Second Order Arithmetic (SOA), Third Order Arithmetic and Fourth Order Arithmetic.

Throughout this paper whenever we write  $X \prec H_\kappa$  and  $\gamma \in X$ ,  $\bar{\gamma}$  always denotes the image of  $\gamma$  under the transitive collapse of  $X$ . If  $U$  is an ultrafilter on  $\kappa$ , we say that  $U$  is countably complete if and only if whenever  $Y \subseteq U$  is countable, we have that  $\bigcap Y \neq \emptyset$ . The distinction between  $V$ -cardinals and  $L$ -cardinals is present throughout the article. Whenever we write  $\omega_n$  (for some  $n$ ) without a superscript it is understood that we mean the  $\omega_n$  of  $V$ . In this paper,  $\kappa$ -model is a model in the form  $L[U]$  such that  $\langle L[U], \in, U \rangle \models U$  is a normal ultrafilter over  $\kappa$ .

## 2. CHARACTERIZATIONS OF THE STRONG REFLECTING PROPERTY FOR $L$ -CARDINALS

In this section we develop the full theory of the strong reflecting property for  $L$ -cardinals and characterize  $SRP^L(\omega_n)$  for  $n \in \omega$ . We also generalize some results on  $SRP^L(\gamma)$  to  $SRP^M(\gamma)$  for any inner model  $M$ .

Recall that an inner model  $M$  is  $L$ -like if  $M$  is in the form  $\langle L[\vec{E}], \in, \vec{E} \rangle$  where  $\vec{E}$  is a coherent sequence of extenders; moreover, for an  $L$ -like inner model  $M$ ,  $M|\theta$  is of the form  $\langle J_{\vec{E}}^\theta, \in, \vec{E} \restriction \theta, \emptyset \rangle$ .<sup>2</sup>

**Convention.** Throughout, whenever we consider an inner model  $M$  we assume that  $M$  is  $L$ -like and has the property that  $M|\theta$  is definable in  $H_\theta$  for any regular cardinal  $\theta > \omega_2$ .<sup>3</sup>

**Definition 2.1.** Let  $\gamma \geq \omega_1$  be an  $L$ -cardinal.

- (i)  $\gamma$  has the strong reflecting property for  $L$ -cardinals, denoted  $SRP^L(\gamma)$ , if and only if for some regular cardinal  $\kappa > \gamma$ , if  $X \prec H_\kappa$ ,  $|X| = \omega$  and  $\gamma \in X$ , then  $\bar{\gamma}$  is an  $L$ -cardinal.
- (ii)  $\gamma$  has the weak reflecting property for  $L$ -cardinals, denoted  $WRP^L(\gamma)$ , if and only if for some regular cardinal  $\kappa > \gamma$ , there is  $X \prec H_\kappa$  such that  $|X| = \omega$ ,  $\gamma \in X$  and  $\bar{\gamma}$  is an  $L$ -cardinal.

**Proposition 2.2.** Suppose  $\gamma \geq \omega_1$  is an  $L$ -cardinal. Then the following are equivalent:

- (1)  $SRP^L(\gamma)$ .
- (2) For any regular cardinal  $\kappa > \gamma$ , if  $X \prec H_\kappa$ ,  $|X| = \omega$  and  $\gamma \in X$ , then  $\bar{\gamma}$  is an  $L$ -cardinal.
- (3) For some regular cardinal  $\kappa > \gamma$ ,  $\{X \mid X \prec H_\kappa, |X| = \omega, \gamma \in X \text{ and } \bar{\gamma} \text{ is an } L\text{-cardinal}\}$  contains a club.
- (4) There exists  $F : \gamma^{<\omega} \rightarrow \gamma$  such that if  $X \subseteq \gamma$  is countable and closed under  $F$ ,<sup>4</sup> then  $\text{o.t.}(X)$  is an  $L$ -cardinal.

<sup>1</sup> $ZFC^-$  denotes  $ZFC$  with the Power Set Axiom deleted and Collection instead of Replacement. For the discussion of the theory  $ZFC$  without power set, see [6].

<sup>2</sup>For the definition of coherent sequences of extenders  $\vec{E}$ ,  $J_\alpha^\vec{E}$  and  $\vec{E} \restriction \alpha$ , see Section 2.2 in [16].

<sup>3</sup>All known core models satisfy this convention.

<sup>4</sup>In this paper, we say that  $X$  is closed under  $F$  if  $F''X^{<\omega} \subseteq X$ .

(5) For any regular cardinal  $\kappa > \gamma$ ,  $\{X \mid X \prec H_\kappa, |X| = \omega, \gamma \in X \text{ and } \bar{\gamma} \text{ is an } L\text{-cardinal}\}$  contains a club.

*Proof.* Note that (2)  $\Rightarrow$  (1), (1)  $\Rightarrow$  (3), (2)  $\Rightarrow$  (5) and (5)  $\Rightarrow$  (3). It suffices to show that (4)  $\Rightarrow$  (2) and (3)  $\Rightarrow$  (4). For the proof see [1, Proposition 2.7].  $\square$

Suppose  $\gamma \geq \omega_1$  is an  $L$ -cardinal. Let (1)\*, (2)\*, (3)\*, (4)\* and (5)\* respectively be the statements which replace “is an  $L$ -cardinal” with “is not an  $L$ -cardinal” in Definition 2.1(i) and statements (2), (3), (4) and (5) in Proposition 2.2. The following corollary is an observation from the proof of Proposition 2.2.

**Corollary 2.3.** (1)\*  $\Leftrightarrow$  (2)\*  $\Leftrightarrow$  (3)\*  $\Leftrightarrow$  (4)\*  $\Leftrightarrow$  (5)\*.

**Proposition 2.4.** Suppose  $\gamma \geq \omega_1$  is an  $L$ -cardinal,  $\kappa$  is regular and  $|\gamma| = \kappa$ . Then the following are equivalent:

- (a)  $SRP^L(\gamma)$ .
- (b) For any bijection  $\pi : \kappa \rightarrow \gamma$ , there exists a club  $D \subseteq \kappa$  such that for any  $\theta \in D$ ,  $\text{o.t.}(\{\pi(\alpha) \mid \alpha < \theta\})$  is an  $L$ -cardinal.
- (c) For some bijection  $\pi : \kappa \rightarrow \gamma$ , there exists a club  $D \subseteq \kappa$  such that for any  $\theta \in D$ ,  $\text{o.t.}(\{\pi(\alpha) \mid \alpha < \theta\})$  is an  $L$ -cardinal.

*Proof.* The proof is essentially the same as the case  $\kappa = \omega_1$  in [1, Proposition 2.9].  $\square$

Let (6)\* and (7)\* respectively be the statement which replaces “is an  $L$ -cardinal” with “is not an  $L$ -cardinal” in Proposition 2.4(b) and Proposition 2.4(c). The following corollary is an observation from the proof of Proposition 2.4.

**Corollary 2.5.** Suppose  $\gamma \geq \omega_1$  is an  $L$ -cardinal,  $\kappa$  is regular and  $|\gamma| = \kappa$ . Then (1)\*  $\Leftrightarrow$  (6)\*  $\Leftrightarrow$  (7)\*.

**Proposition 2.6.** Suppose  $\gamma \geq \omega_1$  is an  $L$ -cardinal. Then the following are equivalent:

- (a)  $WRP^L(\gamma)$ .
- (b) For any regular cardinal  $\kappa > \gamma$ , there is  $X \prec H_\kappa$  such that  $|X| = \omega, \gamma \in X$  and  $\bar{\gamma}$  is an  $L$ -cardinal.
- (c) For some regular cardinal  $\kappa > \gamma$ ,  $\{X \mid X \prec H_\kappa, |X| = \omega, \gamma \in X \text{ and } \bar{\gamma} \text{ is an } L\text{-cardinal}\}$  is stationary.
- (d) For any  $F : \gamma^{<\omega} \rightarrow \gamma$ , there exists  $X \subseteq \gamma$  such that  $X$  is countable, closed under  $F$  and  $\text{o.t.}(X)$  is an  $L$ -cardinal.
- (e) For any regular cardinal  $\kappa > \gamma$ ,  $\{X \mid X \prec H_\kappa, |X| = \omega, \gamma \in X \text{ and } \bar{\gamma} \text{ is an } L\text{-cardinal}\}$  is stationary.

*Proof.* Note that (e)  $\Rightarrow$  (c) and (c)  $\Rightarrow$  (a). It suffices to show that (a)  $\Rightarrow$  (d), (d)  $\Rightarrow$  (b) and (b)  $\Rightarrow$  (e). (a)  $\Rightarrow$  (d) follows from (4)\*  $\Leftrightarrow$  (2)\* in Corollary 2.3. (d)  $\Rightarrow$  (b) follows from (1)\*  $\Leftrightarrow$  (4)\* in Corollary 2.3. (b)  $\Rightarrow$  (e) follows from (3)\*  $\Leftrightarrow$  (1)\* in Corollary 2.3.  $\square$

**Proposition 2.7.** Suppose  $\gamma \geq \omega_1$  is an  $L$ -cardinal,  $\kappa$  is regular and  $|\gamma| = \kappa$ . Then the following are equivalent:

- (1)  $WRP^L(\gamma)$ .
- (2) For some bijection  $\pi : \kappa \rightarrow \gamma$ , there exists a stationary  $D \subseteq \kappa$  such that for any  $\theta \in D$ ,  $\text{o.t.}(\{\pi(\alpha) \mid \alpha < \theta\})$  is an  $L$ -cardinal.

(3) For any bijection  $\pi : \kappa \rightarrow \gamma$ , there exists a stationary  $D \subseteq \kappa$  such that for any  $\theta \in D$ ,  $\text{o.t.}(\{\pi(\alpha) \mid \alpha < \theta\})$  is an  $L$ -cardinal.

*Proof.* Follows from Corollary 2.5 and  $(1)^* \Leftrightarrow (2)^*$  in Corollary 2.3. The proof is standard and we omit the details.  $\square$

**Proposition 2.8.** *The following are equivalent:*

- (1)  $\omega_1$  is a limit cardinal in  $L$ .
- (2)  $WRP^L(\omega_1)$ .
- (3)  $SRP^L(\omega_1)$ .

*Proof.* It suffices to show that  $(1) \Rightarrow (3)$  and  $(2) \Rightarrow (1)$  since  $(3) \Rightarrow (2)$  is immediate.

$(1) \Rightarrow (3)$  Suppose  $\omega_1$  is a limit cardinal in  $L$ . Then  $\{\alpha < \omega_1 : \alpha \text{ is an } L\text{-cardinal}\}$  is a club. By Proposition 2.4,  $SRP^L(\omega_1)$  holds.

$(2) \Rightarrow (1)$  Suppose  $WRP^L(\omega_1)$  holds. Then  $\{X \cap \omega_1 \mid X \prec H_{\omega_2} \wedge |X| = \omega \wedge \text{o.t.}(X \cap \omega_1) \text{ is an } L\text{-cardinal}\}$  is stationary in  $\omega_1$ . It is easy to see that for any  $\alpha < \omega_1$  there is  $\alpha < \beta < \omega_1$  such that  $\beta$  is an  $L$ -cardinal.  $\square$

**Proposition 2.9.** *Suppose  $\gamma \geq \omega_1$  is an  $L$ -cardinal,  $\kappa > \gamma$  is a regular cardinal and  $SRP^L(\gamma)$  holds. If  $Z \prec H_\kappa$ ,  $|Z| \leq \omega_1$  and  $\gamma \in Z$ , then  $\bar{\gamma}$  is an  $L$ -cardinal.*

*Proof.* Suppose  $\bar{\gamma}$  is not an  $L$ -cardinal. Let  $M$  be the transitive collapse of  $Z$  and  $\pi : M \prec H_\kappa$  be the inverse of the collapsing map. Take  $Y \prec H_\kappa$  such that  $|Y| = \omega$  and  $M, \bar{\gamma} \in Y$ . Note that  $Y \models \text{"}\bar{\gamma} \text{ is not an } L\text{-cardinal"}$ . Hence  $\bar{\gamma}$  is not an  $L$ -cardinal.<sup>5</sup> Let  $X = \pi(Y \cap M)$ . Since  $\bar{\gamma} \in Y \cap M$  and  $\pi(\bar{\gamma}) = \gamma$ ,  $\gamma \in X$ . Note that  $X \prec Z \prec H_\kappa$  and the image of  $\gamma$  under the transitive collapse of  $X$  is  $\bar{\gamma}$ . By  $SRP^L(\gamma)$ ,  $\bar{\gamma}$  is an  $L$ -cardinal. Contradiction.  $\square$

**Proposition 2.10.** *Suppose  $\omega_1 \leq \gamma_0 < \gamma_1$  are  $L$ -cardinals. Then  $SRP^L(\gamma_1)$  implies  $SRP^L(\gamma_0)$  (respectively  $WRP^L(\gamma_1)$  implies  $WRP^L(\gamma_0)$ ).*

*Proof.* We only show the strong reflecting property case (the argument for the weak reflecting property case is similar). Let  $\kappa > \gamma_1$  be a regular cardinal. It suffices to show if  $X \prec H_\kappa$ ,  $|X| = \omega$  and  $\{\gamma_0, \gamma_1\} \subseteq X$ , then  $\bar{\gamma}_0$  is an  $L$ -cardinal. Note that  $L_{\gamma_1} \models \gamma_0$  is a cardinal. Since  $\gamma_1 \in X$ ,  $L_{\gamma_1} \in X$ . Since  $L_{\gamma_1} = L_{\bar{\gamma}_1}$  and  $L_{\gamma_1} \models \bar{\gamma}_0$  is a cardinal,  $L_{\bar{\gamma}_1} \models \bar{\gamma}_0$  is a cardinal. By  $SRP^L(\gamma_1)$ ,  $\bar{\gamma}_1$  is an  $L$ -cardinal and hence  $\bar{\gamma}_0$  is an  $L$ -cardinal.  $\square$

**Proposition 2.11.** *The following are equivalent:*

- (1)  $SRP^L(\omega_2)$ .
- (2)  $\omega_2$  is a limit cardinal in  $L$  and for any  $L$ -cardinal  $\omega_1 \leq \gamma < \omega_2$ ,  $SRP^L(\gamma)$  holds.
- (3)  $\{\alpha < \omega_2 \mid \alpha \text{ is an } L\text{-cardinal and } SRP^L(\alpha) \text{ holds}\}$  is unbounded in  $\omega_2$ .

*Proof.*  $(1) \Rightarrow (2)$  By Proposition 2.10, it suffices to show  $\omega_2$  is a limit cardinal in  $L$ . Let  $\kappa > \omega_2$  be the regular cardinal that witnesses  $SRP^L(\omega_2)$ . Fix  $\alpha < \omega_2$ . Pick  $Z \prec H_\kappa$  such that  $|Z| = \omega_1$ ,  $\alpha \subseteq Z$  and  $\omega_2 \in Z$ . By Proposition 2.9,  $\bar{\omega}_2$  is an  $L$ -cardinal. Note that  $\alpha \leq \bar{\omega}_2 < \omega_2$ .

$(2) \Rightarrow (1)$  Suppose  $\kappa > \omega_2$  is a regular cardinal,  $X \prec H_\kappa$ ,  $|X| = \omega$  and  $\omega_2 \in X$ . We show that  $\bar{\omega}_2$  is an  $L$ -cardinal. Note that  $\bar{\omega}_2 = \text{o.t.}(X \cap \omega_2)$ . Let  $E = \{\gamma \mid \omega_1 \leq \gamma < \omega_2 \wedge \gamma \text{ is an } L\text{-cardinal}\}$ .  $E$  is definable in  $H_\kappa$ . Since  $\omega_2$  is a limit cardinal in  $L$ ,

<sup>5</sup> $\bar{\gamma}$  is the image of  $\bar{\gamma}$  under the transitive collapse of  $Y$ .

$E$  is cofinal in  $\omega_2$  and hence  $E \cap X$  is cofinal in  $\omega_2 \cap X$ . For  $\gamma \in E \cap X$ ,  $\bar{\gamma} = o.t.(X \cap \gamma)$  and by  $SRP^L(\gamma)$ ,  $\bar{\gamma}$  is an  $L$ -cardinal. Note that  $\bar{\omega}_2 = \sup(\{\bar{\gamma} \mid \gamma \in E \cap X\})$ . Hence  $\bar{\omega}_2$  is an  $L$ -cardinal.

(1)  $\Leftrightarrow$  (3) Follows from (1)  $\Leftrightarrow$  (2) and Proposition 2.10.  $\square$

The notion of remarkable cardinal is introduced by Ralf Schindler in [15]. Any remarkable cardinal is remarkable in  $L$  (cf.[15, Lemma 1.7]).

**Definition 2.12.** ([15])

- (1) Let  $\kappa$  be a cardinal,  $G$  be  $Col(\omega, < \kappa)$ -generic over  $V$ ,  $\theta > \kappa$  be a regular cardinal and  $X \in [H_\theta^{V[G]}]^\omega$ . We say that  $X$  condenses remarkably if  $X = \text{ran}(\pi)$  for some elementary  $\pi : (H_\beta^{V[G \cap H_\alpha^V]}, \in, H_\beta^V, G \cap H_\alpha^V) \rightarrow (H_\theta^{V[G]}, \in, H_\theta^V, G)$  where  $\alpha = \text{crit}(\pi) < \beta < \kappa$  and  $\beta$  is a cardinal in  $V$ .
- (2) For regular cardinal  $\theta > \kappa$ ,  $\kappa$  is  $\theta$ -remarkable if and only if in  $V^{Col(\omega, < \kappa)}$ ,  $\{X \in [H_\theta]^\omega : X \text{ condenses remarkably}\}$  is stationary. We say that  $\kappa$  is remarkable if  $\kappa$  is  $\theta$ -remarkable for all regular cardinal  $\theta > \kappa$ .

**Lemma 2.13.** ([1, Lemma 2.3]) *Suppose  $\kappa$  is an  $L$ -cardinal. The following are equivalent:*

- (1)  $\kappa$  is remarkable in  $L$ ;
- (2) If  $\gamma \geq \kappa$  is an  $L$ -cardinal,  $\theta > \gamma$  is a regular cardinal in  $L$ , then  $\Vdash_{Col(\omega, < \kappa)}^L$  “ $\{X \mid X \prec L_\theta[\dot{G}], |X| = \omega \text{ and } o.t.(X \cap \dot{\gamma}) \text{ is an } L\text{-cardinal}\}$  is stationary”.

**Corollary 2.14.** *If  $\kappa$  is remarkable in  $L$  and  $G$  is  $Col(\omega, < \kappa)$ -generic over  $L$ , then  $L[G] \models WRP^L(\gamma)$  holds for any  $L$ -cardinal  $\gamma \geq \kappa$ .*

*Proof.* Follows from Lemma 2.13.  $\square$

Fix some  $L$ -cardinal  $\gamma \geq \omega_1$ .  $SRP^L(\gamma)$  is upward absolute (cf. [1, Proposition 2.11]).<sup>6</sup> As a corollary,  $WRP^L(\gamma)$  is downward absolute.<sup>7</sup> So if  $WRP^L(\gamma)$  holds, then  $WRP^L(\gamma)$  holds in  $L$ . The converse is not true in general.

**Proposition 2.15.** *Suppose  $WRP^L(\kappa)$  holds where  $\kappa \geq \omega_1$  is an  $L$ -cardinal. Then  $L \models \omega_1$  is  $\kappa^+$ -remarkable and for any regular  $\theta > \kappa$  in  $L$ ,  $L \models \omega_1$  is  $\theta$ -remarkable.*

*Proof.*  $L \models WRP^L(\kappa)$  iff  $\{X \mid X \prec L_{\kappa^+}, |X| = \omega \text{ and } o.t.(X \cap \kappa) \text{ is an } L\text{-cardinal}\}$  is stationary in  $L$  iff for any  $L$ -regular cardinal  $\theta > \kappa$ ,  $\{X \mid X \prec L_\theta, |X| = \omega \text{ and } o.t.(X \cap \kappa) \text{ is an } L\text{-cardinal}\}$  is stationary in  $L$ . For  $L$ -regular cardinal  $\theta > \kappa$ ,  $L \models \omega_1$  is  $\theta$ -remarkable iff for any  $G$  which is  $Col(\omega, < \omega_1)$ -generic over  $L$ ,  $L[G] \models \{X \in [L_\theta]^\omega \mid X = \text{ran}(\pi), \pi : (L_\beta[G \restriction \alpha], \in, L_\beta, G \restriction \alpha) \prec (L_\theta[G], \in, L_\theta, G) \text{ where } \alpha = \text{crit}(\pi) < \beta < \omega_1 \text{ and } \beta \text{ is an } L\text{-cardinal}\}$  is stationary. Note that  $L \models WRP^L(\kappa)$  and  $Col(\omega, < \omega_1)$  is stationary preserving.  $\square$

**Corollary 2.16.** *“For any  $L$ -cardinal  $\gamma \geq \omega_1$ ,  $WRP^L(\gamma)$  holds” is equiconsistent with  $\omega_1$  is remarkable.*

*Proof.* Follows from Corollary 2.14 and Proposition 2.15.  $\square$

**Theorem 2.17.** (Set forcing) *The following two theories are equiconsistent:*

- (1)  $SRP^L(\omega_2)$ .

<sup>6</sup>The key point is that the statement Proposition 2.2(4) is upward absolute.

<sup>7</sup>The key point is that the statement Proposition 2.6(d) is downward absolute.

(2)  $ZFC +$  there exists a remarkable cardinal with a weakly inaccessible cardinal above it.

*Proof.* We first show that the consistency of (2) implies the consistency of (1). Let  $S = \{\omega_1 \leq \alpha < \omega_2 \mid \alpha \text{ is an } L\text{-cardinal}\}$ . Note that  $SRP^L(\omega_2)$  is equivalent to  $S$  being a club such that  $SRP^L(\alpha)$  holds for any  $\alpha \in S$ . In [1, Section 3.1], assuming there exists a remarkable cardinal with a weakly inaccessible cardinal above it, we force a model  $L[G, H]$  in which  $S$  is a club and  $SRP^L(\alpha)$  holds for any  $\alpha \in S$ . So  $SRP^L(\omega_2)$  holds in  $L[G, H]$ .

From [1, Section 3.2-3.4], if  $S$  is a club and  $SRP^L(\alpha)$  holds for any  $\alpha \in S$ , then we can force a model of  $Z_3 + HP(L)$ . So the consistency of (1) implies the consistency of  $Z_3 + HP(L)$ . By [2, Theorem 3.2],  $Z_3 + HP(L)$  implies  $L \models ZFC + \omega_1^V$  is remarkable. By Proposition 2.11,  $\omega_2^V$  is inaccessible in  $L$ . So the consistency of (1) implies the consistency of (2).  $\square$

**Definition 2.18.** Suppose  $M$  is an inner model and  $\gamma \geq \omega_1$  is an  $M$ -cardinal. We say that  $\gamma$  has the strong reflecting property for  $M$ -cardinals, denoted  $SRP^M(\gamma)$ , if and only if for some regular cardinal  $\kappa > \gamma$ , if  $X \prec H_\kappa$ ,  $|X| = \omega$  and  $\gamma \in X$ , then  $\bar{\gamma}$  is an  $M$ -cardinal.

**Definition 2.19.** Suppose  $M$  is an inner model. We say that  $M$  has the full covering property if for any set  $X$  of ordinals, there is  $Y \in M$  such that  $X \subseteq Y$  and  $|Y| = |X| + \omega_1$ . We say that  $M$  has the rigidity property if there is no nontrivial elementary embedding from  $M$  to  $M$ .

**Theorem 2.20.** Suppose  $M$  is an inner model which satisfies Convention 2 and has both the full covering and the rigidity property. Then, for every  $M$ -cardinal  $\gamma > \omega_2$ ,  $SRP^M(\gamma)$  fails.

*Proof.* Suppose  $SRP^M(\gamma)$  holds for some  $\gamma > \omega_2$ . Let  $\kappa > \gamma$  be the witnessing regular cardinal for  $SRP^M(\gamma)$ . Build an elementary chain  $\langle Z_\alpha \mid \alpha < \omega_1 \rangle$  of submodels of  $H_\kappa$  such that for all  $\alpha < \beta < \omega_1$ ,  $Z_\alpha \prec Z_\beta \prec H_\kappa$ ,  $Z_\alpha \in Z_\beta$ ,  $|Z_\alpha| = \omega$  and  $\{\gamma, \omega_2\} \subseteq Z_0$ .

Let  $Z = \bigcup_{\alpha < \omega_1} Z_\alpha$ . Then  $|Z| = \omega_1$  and  $Z \prec H_\kappa$ . Let  $\pi : N \cong Z \prec H_\kappa$  and  $\pi_\alpha : N_\alpha \cong Z_\alpha \prec H_\kappa$  be the inverses of the collapsing maps. Let  $j_\alpha : N_\alpha \prec N$  be the induced elementary embedding. Since  $\omega_1 \subseteq Z$ ,  $\text{crit}(\pi) > \bar{\omega}_1$ . Since  $\omega_2 \in Z$  and  $|Z| = \omega_1$ ,  $\text{crit}(\pi) \leq \bar{\omega}_2$ . So  $\text{crit}(\pi) = \bar{\omega}_2$ .

Note that Proposition 2.9 still holds if we replace  $L$  with  $M$ . By  $SRP^M(\gamma)$ ,  $\bar{\gamma}$  is an  $M$ -cardinal. Since  $M \restriction \bar{\gamma}$  is definable in  $H_\kappa$ ,  $\mathcal{P}(\bar{\omega}_2) \cap M \subseteq M \restriction \bar{\gamma} \in N$  and  $\mathcal{P}(\bar{\omega}_2) \cap M \in N$ . Define  $U = \{X \subseteq \bar{\omega}_2 \mid X \in M \wedge \bar{\omega}_2 \in \pi(X)\}$ .  $U$  is an  $M$ -ultrafilter. For  $\alpha < \omega_1$ , the image of  $Z_\alpha$  under the transitive collapse of  $Z$  is  $j_\alpha "N_\alpha$  and  $j_\alpha "N_\alpha \in N$ .

**Lemma 2.21.**  $U$  is countably complete.

*Proof.* Suppose  $Y \subseteq U$  and  $Y$  is countable. We show that  $\bigcap Y \neq \emptyset$ . Since  $Y \subseteq N$ , take  $\alpha < \omega_1$  large enough such that  $Y \subseteq j_\alpha "N_\alpha$ . Let  $S = \mathcal{P}(\bar{\omega}_2) \cap M \cap j_\alpha "N_\alpha$ . Note that  $S \in N$  and  $N \models S$  is countable.

Note that  $H_\kappa \models "M \text{ has the full covering property}"^8$  and hence  $N \models M$  has the full covering property. Fix  $T \in N$  such that  $T \subseteq \mathcal{P}(\bar{\omega}_2) \cap M$ ,  $T \supseteq S$ ,  $T \in M$  and  $N \models |T| = \omega_1$ . Since  $\bar{\omega}_2 = \text{crit}(\pi) > \omega_1$ ,  $\pi(T) = \pi "T$ . Since  $T \in N$ ,  $\pi(T) \cap M \in N$ .

<sup>8</sup>Here we use that  $M \restriction \theta$  is definable in  $H_\theta$  for regular cardinal  $\theta > \omega_2$ .

**Claim 2.22.**  $U \cap T \in N$ .

*Proof.* Since  $\pi(T) = \pi^{\text{``}}T \in M$ ,  $\pi^{\text{``}}(U \cap T) = \{\pi(A) \mid A \in T \wedge \bar{\omega}_2 \in \pi(A)\} = \{B \in \pi(T) \mid \bar{\omega}_2 \in B\}$  and  $\pi^{\text{``}}(U \cap T) \in M$ . Note that  $\mathcal{P}(\pi^{\text{``}}T) \cap M = \pi^{\text{``}}(\mathcal{P}(T) \cap M)$  since for all  $D \in \mathcal{P}(T) \cap M$ ,  $\pi(D) = \pi^{\text{``}}D$ . Since  $\pi^{\text{``}}(U \cap T) \in \mathcal{P}(\pi^{\text{``}}T) \cap M$ ,  $\pi^{\text{``}}(U \cap T) = \pi(D) = \pi^{\text{``}}D$  for some  $D \in \mathcal{P}(T) \cap M \subseteq N$ . So  $U \cap T = D$  and hence  $U \cap T \in N$ .  $\square$

Note that  $Y \subseteq j_\alpha^{\text{``}}N_\alpha \cap \mathcal{P}(\bar{\omega}_2) \cap M = S \subseteq T$ . Since  $Y \subseteq T \cap U$ , to show that  $\bigcap Y \neq \emptyset$ , it suffices to show that  $\bigcap (U \cap T) \neq \emptyset$ . Note that  $\bar{\omega}_2 \in \bigcap \pi^{\text{``}}(U \cap T)$  and  $\pi(U \cap T) = \pi^{\text{``}}(U \cap T)$ . Then  $\bigcap \pi^{\text{``}}(U \cap T) = \bigcap \pi(U \cap T) = \pi(\bigcap (U \cap T)) \neq \emptyset$ . So  $\bigcap (U \cap T) \neq \emptyset$ .  $\square$

So we can build a nontrivial embedding from  $M$  to  $M$  which contradicts the rigidity property of  $M$ .  $\square$

**Theorem 2.23.** *The following are equivalent:*

- (i)  $SRP^L(\gamma)$  holds for some  $L$ -cardinal  $\gamma > \omega_2$ .
- (ii)  $0^\sharp$  exists.
- (iii)  $SRP^L(\gamma)$  holds for every  $L$ -cardinal  $\gamma \geq \omega_1$ .

*Proof.* (i)  $\Rightarrow$  (ii) Assume  $0^\sharp$  does not exist. Then  $L$  satisfies all the conditions for  $M$  in Theorem 2.20. From the proof of Theorem 2.20 (replace  $M$  with  $L$ ),  $SRP^L(\gamma)$  does not hold for any  $L$  cardinal  $\gamma > \omega_2$ .

(ii)  $\Rightarrow$  (iii) Note that if  $X \prec H_\kappa$  and  $\gamma \in X$ , then  $\mathcal{M}(0^\sharp, \gamma + 1) \in X$  and its image under the transitive collapse of  $X$  is  $\mathcal{M}(0^\sharp, \bar{\gamma} + 1)$ .<sup>9</sup> Note that for  $\alpha \in \text{Ord}$ ,  $\mathcal{M}(0^\sharp, \alpha) \prec L$ .  $\square$

So for  $n \geq 3$ ,  $SRP^L(\omega_n)$  is equivalent to  $0^\sharp$  exists. We have characterized  $SRP^L(\omega_n)$  for  $n \geq 1$ .

**Definition 2.24.** Suppose  $M$  is an inner model. For  $M$ -cardinal  $\lambda$ , let  $SRP_{<\lambda}^M(\lambda)$  denote the statement: for some regular cardinal  $\theta > \lambda$ , if  $X \prec H_\theta$ ,  $|X| < \lambda$  and  $\lambda \in X$ , then  $\bar{\lambda}$  is an  $M$ -cardinal.

**Fact 2.25.** ([13, Theorem 1.3]) Assume  $0^\dagger$  does not exist but there is an inner model with a measurable cardinal and  $L[U]$  is chosen such that  $\kappa = \text{crit}(U)$  is as small as possible. The one of the following holds:

- (a) For every set  $X$  of ordinals, there is a set  $Y \in L[U]$  such that  $Y \supseteq X$  and  $|Y| = |X| + \omega_1$ ;
- (b) There is a sequence  $C \subseteq \kappa$ , which is Prikry generic over  $L[U]$ , such that for all set  $X$  of ordinals, there is a set  $Y \in L[U, C]$  such that  $Y \supseteq X$  and  $|Y| = |X| + \omega_1$ .

**Fact 2.26.** ([10, 21.22 Exercise]) The following are equivalent:

- (1)  $0^\dagger$  exists.
- (2) There is a  $\kappa$ -model for some  $\kappa$  and an elementary embedding from that model to itself with critical point greater than  $\kappa$ .

**Theorem 2.27.** *Suppose there is an inner model with a measurable cardinal and  $L[U]$  is chosen such that  $\kappa = \text{crit}(U)$  is as small as possible. Suppose  $\lambda > \kappa^+$  is an  $L[U]$ -cardinal. Then  $SRP_{<\lambda}^{L[U]}(\lambda)$  if and only if  $0^\dagger$  exists.*

<sup>9</sup> $\mathcal{M}(0^\sharp, \alpha)$  is the unique transitive  $(0^\sharp, \alpha)$ -model. For the notation, see [10].

*Proof.* ( $\Rightarrow$ ) We assume that  $0^\dagger$  does not exist and try to get a contradiction. By Fact 2.25, we need to discuss two cases.

Case 1: Fact 2.25(a) holds. Let  $\theta > \lambda$  be the witness regular cardinal for  $SRP_{<\lambda}^{L[U]}(\lambda)$ . Build an elementary chain  $\langle Z_\alpha \mid \alpha < \kappa \rangle$  of submodels of  $H_\theta$  such that for  $\alpha < \beta < \kappa$ ,  $Z_\alpha \prec Z_\beta \prec H_\theta$ ,  $Z_\alpha \in Z_\beta$ ,  $|Z_\alpha| = \kappa$  and  $\{\kappa^+, \lambda\} \cup \text{tr}(\{U\}) \subseteq Z_0$ .<sup>10</sup> Let  $Z = \bigcup_{\alpha < \kappa} Z_\alpha$ . Then  $|Z| = \kappa$ . Let  $\pi : N \cong Z \prec H_\theta$  and  $\pi_\alpha : N_\alpha \cong Z_\alpha \prec H_\theta$  be the inverses of the collapsing maps. Since  $Z_\alpha \prec Z$ , let  $j_\alpha : N_\alpha \prec N$  be the induced embedding. Then  $\pi_\alpha = \pi \circ j_\alpha$  and  $N = \bigcup_{\alpha < \kappa} j_\alpha "N_\alpha$ . Let  $\text{crit}(\pi) = \eta$ . Then  $\eta > \kappa = \bar{\kappa}$  and since  $|Z| = \kappa$ ,  $\eta \leq \kappa^+$ . So  $\eta = \kappa^+ < \bar{\lambda}$ . By  $SRP_{<\lambda}^{L[U]}(\lambda)$ ,  $\bar{\lambda}$  is an  $L[U]$ -cardinal. Let  $W = \{X \subseteq \eta \mid X \in L[U] \text{ and } \eta \in \pi(X)\}$ . Note that  $U = \bar{U} \in N$  and  $W \subseteq L_{\bar{\lambda}}[U] \subseteq N$ .  $W$  is  $L[U]$ -ultrafilter on  $\eta$ . Note that  $Z \models "|Z_\alpha| = \kappa"$  and the image of  $Z_\alpha$  under the transitive collapse of  $Z$  is  $j_\alpha "N_\alpha$ . So for  $\alpha < \kappa$ ,  $j_\alpha "N_\alpha \in N$  and  $N \models "|j_\alpha "N_\alpha| = \kappa"$ .

**Lemma 2.28.**  *$W$  is countably complete.*

*Proof.* Suppose  $Y \subseteq W$  and  $Y$  is countable. We show that  $\bigcap Y \neq \emptyset$ . Since  $Y \subseteq N$ , take  $\alpha < \kappa$  large enough such that  $Y \subseteq j_\alpha "N_\alpha$ . Let  $S = \mathcal{P}(\eta) \cap L[U] \cap j_\alpha "N_\alpha$ . Note that  $\mathcal{P}(\eta) \cap L[U] \in N$  and hence  $S \in N$ .  $N \models |S| \leq \kappa$ . Since Fact 2.25(a) holds in  $H_\theta$  and  $N \prec H_\theta$ , Fact 2.25(a) holds in  $N$ . Take  $T \in N$  such that  $T \subseteq \mathcal{P}(\eta) \cap L[U]$ ,  $T \supseteq S$ ,  $T \in L[U]$  and  $N \models |T| \leq \kappa$ . Since  $\eta > \kappa$ ,  $\pi(T) = \pi "T$ . Let  $\bar{T} = \{X \in T \mid \eta \in \pi(X)\}$ .

**Claim 2.29.**  $\bar{T} \in N$ .

*Proof.* Since  $N \models |T| \leq \kappa$ , there is  $h \in N$  such that  $h : T \leftrightarrow \gamma$  for some  $\gamma < \eta$ . Then  $\bar{T} = \{X \in T \mid \eta \in \pi((h^{-1})(h(X)))\}$ . So  $\bar{T} \in N$ .  $\square$

Note that  $\bigcap \bar{T} \neq \emptyset$  since  $\pi(\bar{T}) = \pi " \bar{T}$  and  $\eta \in \bigcap \pi " \bar{T} = \bigcap \pi(\bar{T}) = \pi(\bigcap \bar{T})$ . Since  $Y \subseteq S \subseteq T$  and  $Y \subseteq W$ ,  $Y \subseteq \bar{T}$  and hence  $\bigcap Y \neq \emptyset$ .  $\square$

So there exists a nontrivial elementary embedding  $j : L[U] \prec L[U]$  with  $\text{crit}(j) = \eta > \kappa$ . By Fact 2.26,  $0^\dagger$  exists. Contradiction.

Case 2: Fact 2.25(b) holds. The proof is essentially the same as Case 1 with small modifications (for example, let  $\text{tr}(\{U, C\}) \subseteq Z_0$  and  $W = \{X \subseteq \eta \mid X \in L[U, C] \text{ and } \eta \in \pi(X)\}$ ). Since Prikry forcing preserves all cardinals,  $\bar{\lambda}$  is an  $L[U, C]$ -cardinal. As in Case 1, we can show that there exists a nontrivial elementary embedding  $j : L[U, C] \prec L[U, C]$ . Since  $j(U, C) = (U, C)$ ,  $j \upharpoonright L[U] : L[U] \prec L[U]$ .  $\text{crit}(j \upharpoonright L[U]) = \eta > \kappa$ . So by Fact 2.26,  $0^\dagger$  exists. Contradiction.

( $\Leftarrow$ ) Assume  $0^\dagger$  exists. Suppose  $\theta > \lambda$  is regular,  $X \prec H_\theta$ ,  $|X| < \lambda$  and  $\lambda \in X$ . We show that  $\bar{\lambda}$  is an  $L[U]$ -cardinal. Since  $\lambda \in X$  and  $0^\dagger \in X$ ,  $\mathcal{M}(0^\dagger, \omega, \lambda + 1) \in X$ .<sup>11</sup> Note that for any  $\alpha, \beta \in \text{Ord}$ ,  $\mathcal{M}(0^\dagger, \alpha, \beta) \prec L[U]$ . Since  $\lambda$  is an  $L[U]$ -cardinal and  $\lambda \in \mathcal{M}(0^\dagger, \omega, \lambda + 1)$ ,  $\mathcal{M}(0^\dagger, \omega, \lambda + 1) \models \lambda$  is a cardinal. Note that the image of  $\mathcal{M}(0^\dagger, \omega, \lambda + 1)$  under the transitive collapse of  $X$  is  $\mathcal{M}(0^\dagger, \omega, \bar{\lambda} + 1)$ . So  $\mathcal{M}(0^\dagger, \omega, \bar{\lambda} + 1) \models "\bar{\lambda} \text{ is a cardinal}"$ . Since  $\mathcal{M}(0^\dagger, \omega, \bar{\lambda} + 1) \prec L[U]$ ,  $\bar{\lambda}$  is an  $L[U]$ -cardinal.  $\square$

<sup>10</sup>In this article,  $\text{tr}(X)$  stands for the transitive closure of  $X$ .

<sup>11</sup>Note that  $\mathcal{M}(0^\dagger, \omega, \alpha)$  is the unique transitive  $(0^\dagger, \omega, \alpha)$ -model. For the notation of  $\mathcal{M}(0^\dagger, \omega, \alpha)$ , see [10].



In [14], Thoralf R  sch and Ralf Schindler introduced the condensation principle  $\nabla_\kappa$ : for any regular cardinal  $\theta > \kappa$ ,  $\{X \prec L_\theta \mid |X| < \kappa, X \cap \kappa \in \kappa \text{ and } L \models o.t.(X \cap \theta) \text{ is a cardinal}\}$  is stationary. The notion of the strong reflecting property for  $L$ -cardinals was introduced before the author knew about the work on  $\nabla_\kappa$  in [14]. The following theorem summarizes the strength of  $\nabla_{\omega_n}$  for  $n \in \omega$ .

**Theorem 2.30.** (1) ([14, Theorem 2, 4]) *The following theories are equiconsistent:*

- (a)  $ZFC + \nabla_{\omega_1}$ .
- (b)  $ZFC + \nabla_{\omega_2}$ .
- (c)  $ZFC + \text{there exists a remarkable cardinal}$ .

(2) [14, Corollary 12] *For  $n \geq 3$ ,  $\nabla_{\omega_n}$  is equivalent to  $0^\sharp$  exists.*

Now we discuss the relationship between  $SRP^L(\omega_n)$  and  $\nabla_{\omega_n}$  for  $n \in \omega$ . By Theorem 2.23 and 2.30, for  $n \geq 3$ ,  $SRP^L(\omega_n)$  is equivalent to  $\nabla_{\omega_n}$ . If  $\kappa$  is regular cardinal and  $\nabla_\kappa$  holds, then  $\kappa$  is remarkable in  $L$  (cf. [14, Lemma 7]). By Proposition 2.8,  $\nabla_{\omega_1}$  implies  $SRP^L(\omega_1)$  which is strictly weaker. By Theorem 2.17,  $SRP^L(\omega_2)$  does not imply  $\nabla_{\omega_2}$  since  $\nabla_{\omega_2}$  implies  $\omega_2$  is remarkable in  $L$ . By Theorem 2.30 and 2.17, the strength of  $SRP^L(\omega_2)$  is strictly stronger than  $\nabla_{\omega_2}$ .

In Definition 2.1, we only consider countable elementary submodels of  $H_\kappa$ . Similarly as  $\nabla_\kappa$  we could also consider uncountable elementary submodels of  $H_\kappa$ . However this does not change the picture. Obviously,  $SRP^L_{<\omega_1}(\omega_1)$  iff  $SRP^L(\omega_1)$ . By Proposition 2.9,  $SRP^L_{<\omega_2}(\omega_2)$  iff  $SRP^L(\omega_2)$ . By Theorem 2.20, for  $n \geq 3$ ,  $SRP^L_{<\omega_n}(\omega_n)$  iff  $0^\sharp$  exists iff  $SRP^L(\omega_n)$ .

### 3. HARRINGTON'S PRINCIPLE $HP(L)$ AND ITS GENERALIZATION

In this section, we define the generalized Harrington's Principle  $HP(M)$  for any inner model  $M$ . Considering various known examples of inner models we give particular characterizations of  $HP(M)$ , while we also show that in some cases this generalized principle fails.

Recall that for limit ordinal  $\alpha > \omega$ ,  $\alpha$  is  $x$ -admissible if and only if there is no  $\Sigma_1(L_\alpha[x])$  mapping from an ordinal  $\delta < \alpha$  cofinally into  $\alpha$  (see [4, Lemma 7.2]).

**Definition 3.1.** Suppose  $M$  is an inner model. The Generalized Harrington's Principle  $HP(M)$  denotes the following statement: there is a real  $x$  such that, for any ordinal  $\alpha$ , if  $\alpha$  is  $x$ -admissible then  $\alpha$  is an  $M$ -cardinal, i.e.,  $M \models \alpha$  is a cardinal.  $HP(L)$  denotes Harrington's Principle.

Harrington's principle  $HP(L)$  was isolated by Harrington in the proof of his celebrated theorem " $Det(\Sigma_1^1)$  implies  $0^\sharp$ " in [7].

**Fact 3.2.** (Essentially [4])  $(Z_4)$   $L_{\omega_2}$  has an uncountable set of indiscernibles if and only if  $0^\sharp$  exists.

**Theorem 3.3.**  $(Z_4)$  *The following are equivalent.*<sup>12</sup>

- (1)  $HP(L)$ .
- (2)  $L_{\omega_2}$  has an uncountable set of indiscernibles.
- (3)  $0^\sharp$  exists.

<sup>12</sup>In [2], we define  $0^\sharp$  as the minimal iterable mouse and prove in  $Z_4$  that  $HP(L)$  is equivalent to  $0^\sharp$  exists. Theorem 3.3 proves that these two definitions of  $0^\sharp$  are equivalent in  $Z_4$ .

*Proof.* Note that in  $Z_2, 0^\sharp$  implies  $HP(L)$  since any  $0^\sharp$ -admissible ordinal is an  $L$ -cardinal. It suffices to show that (1)  $\Rightarrow$  (2). Let  $a$  be the witness real for  $HP(L)$ . We work in  $L[a]$ . Pick  $\eta > \omega_2$  and  $N$  such that  $\eta$  is  $a$ -admissible,  $N \prec L_\eta[a]$ ,  $\omega_2 \in N$ ,  $|N| = \omega_1$  and  $N$  is closed under  $\omega$ -sequences. Let  $j : L_\theta[a] \cong N \prec L_\eta[a]$  be the inverse of the collapsing map and  $\kappa = \text{crit}(j)$ . By  $HP(L)$ ,  $\theta$  is an  $L$ -cardinal. Define  $U = \{X \subseteq \kappa \mid X \in L \wedge \kappa \in j(X)\}$ . Note that  $(\kappa^+)^L \leq \theta < \omega_2$  and  $U \subseteq L_\theta$  is an  $L$ -ultrafilter on  $\kappa$ . Do the ultrapower construction for  $\langle L_{\omega_2}, \in, U \rangle$ . Since  $L_\theta[a]$  is closed under  $\omega$ -sequences,  $L_{\omega_2}/U$  is well founded and hence we get a nontrivial elementary embedding  $e : L_{\omega_2} \prec L_{\omega_2}$  with  $\text{crit}(e) = \kappa$ .

Now we show that there exists a club on  $\omega_2$  of regular  $L$ -cardinals. Suppose  $X \prec L_\eta[a]$ ,  $\omega_1 \subseteq X$  and  $\omega_2 \in X$ . The transitive collapse of  $X$  is  $L_{\bar{\eta}}[a]$  for some  $\bar{\eta}$ . Since  $L_\eta \models \omega_2$  is a regular cardinal,  $L_{\bar{\eta}} \models \bar{\omega}_2$  is a regular cardinal. By  $HP(L)$ ,  $\bar{\eta}$  is an  $L$ -cardinal and hence  $\bar{\omega}_2$  is a regular  $L$ -cardinal. Since  $\omega_1 \subseteq X$ ,  $\bar{\omega}_2 = X \cap \omega_2$ . We have shown that if  $X \prec L_\eta[a]$ ,  $\omega_1 \subseteq X$  and  $\omega_2 \in X$ , then  $X \cap \omega_2 = \bar{\omega}_2$  is a regular  $L$ -cardinal. So there exists a club on  $\omega_2$  of regular  $L$ -cardinals. Let  $D$  be such a club such that  $D \cap (\kappa + 1) = \emptyset$ .

**Claim 3.4.** For any  $\alpha \in D$ ,  $e(\alpha) = \alpha$ .

*Proof.* Suppose  $\alpha \in D$  and  $f \in L_{\omega_2}$  where  $f : \kappa \rightarrow \alpha$ . Since  $\alpha > \kappa$  is a regular  $L$ -cardinal,  $f$  is bounded by some  $\eta < \alpha$ . So  $[f] < [c_\eta]$ . Hence  $e(\alpha) = \lim_{\beta \rightarrow \alpha} e(\beta)$ . If  $\beta < \alpha$ , then  $|e(\beta)| \leq (|\beta^\kappa|)^L \leq \alpha$ . So  $e(\alpha) = \alpha$ .  $\square$

We define a sequence  $\langle C_\alpha : \alpha < \omega_1 \rangle$  as follows. Let  $C_0 = D$ . For any  $\nu < \omega_1$ ,  $C_{\nu+1} = \{\mu \in C_\nu \mid \mu \text{ is the } \mu\text{-th element of } C_\nu \text{ in the increasing enumeration of } C_\nu\}$ . If  $\nu \leq \omega_1$  is a limit ordinal,  $C_\nu = \bigcap_{\beta < \nu} C_\beta$ . Note that  $C_\nu$  is a club on  $\omega_2$  for all  $\nu \leq \omega_1$ . By Claim 3.4, for  $\nu \leq \omega_1$ ,  $e \restriction C_\nu = \text{id}$ . Now we will find  $\omega_1$ -many indiscernibles for  $(L_{\omega_2}, \in)$ . The rest of the argument essentially follows from [8, Theorem 18.20].

For each  $\nu < \omega_1$ , let  $M_\nu$  be the Skolem hull of  $\kappa \cup C_\nu$  in  $L_{\omega_2}$ . The transitive collapse of  $M_\nu$  is  $L_{\omega_2}$ . Let  $i_\nu : L_{\omega_2} \cong M_\nu \prec L_{\omega_2}$  be the inverse of the collapsing map and  $\kappa_\nu = i_\nu(\kappa)$ . By [8, Lemma 18.24, 18.25, 18.26],  $\{\kappa_\nu \mid \nu < \omega_1\}$  is a set of indiscernibles for  $L_{\omega_2}$ .<sup>13</sup>  $\square$

**Theorem 3.5.** ([2])  $Z_3 + HP(L)$  does not imply  $0^\sharp$  exists.

By a similar argument as in Theorem 3.3 we can show from  $Z_3 + HP(L)$  that there exists a nontrivial elementary embedding  $j : L_{\omega_1} \prec L_{\omega_1}$  and there is a club  $C \subseteq \omega_1$  of regular  $L$ -cardinals. However, by Theorem 3.5, from these we can not prove in  $Z_3$  that  $0^\sharp$  exists.

Note that Theorem 3.3 still holds if we replace the term “ $L$ -cardinal” with any large cardinal notion compatible with  $L$  in the definition of  $HP(L)$ . This is because the Silver indiscernibles can have any large cardinal property compatible with  $L$ .<sup>14</sup>

**Fact 3.6.** ([10, Theorem 21.15]) The following are equivalent:

- (1)  $0^\dagger$  exists.

<sup>13</sup>Note that the proof of [8, Theorem 18.20], as opposed to the proof of Theorem 3.3 above, is not done in  $Z_4$ .

<sup>14</sup>Examples of large cardinal notions compatible with  $L$ : inaccessible cardinal, reflecting cardinal, Mahlo cardinal, weakly compact, indescribable cardinal, unfoldable cardinal, subtle cardinal, ineffable cardinal, 1-iterable cardinal, remarkable cardinal, 2-iterable cardinal and  $\omega$ -Erdős cardinal.

- (2) For every uncountable cardinal  $\kappa$  there is a  $\kappa$ -model and a double class  $\langle X, Y \rangle$  of indiscernibles for it such that:  $X \subseteq \kappa$  is closed unbounded,  $Y \subseteq \text{Ord} \setminus (\kappa + 1)$  is a closed unbounded class,  $X \cup \{\kappa\} \cup Y$  contains every uncountable cardinal and the Skolem hull of  $X \cup Y$  in the  $\kappa$ -model is again the model.

**Fact 3.7.** ([12, Lemma 1.7]) Suppose that  $A$  is a set,  $X \prec L_\alpha[A]$  where  $\alpha \in \text{Ord} \cup \{\text{Ord}\}$  and the transitive closure of  $A \cap L_\alpha[A]$  is contained in  $X$ . Then  $X \cong L_{\alpha'}[A]$  for some  $\alpha' \leq \alpha$ .

**Fact 3.8.** (Folklore) Suppose  $0^\dagger$  exists,  $L[U]$  is the unique  $\kappa$ -model and  $\langle X, Y \rangle$  is the double class of indiscernibles for  $L[U]$  as in Fact 3.6. If  $\alpha \leq \kappa$  is  $0^\dagger$ -admissible, then  $X$  is unbounded in  $\alpha$ , and if  $\alpha > \kappa$  is  $0^\dagger$ -admissible, then  $Y$  is unbounded in  $\alpha$ .<sup>15</sup>

**Theorem 3.9.** Suppose  $\kappa$  is a measurable cardinal and  $L[U]$  is the unique  $\kappa$ -model. Then  $HP(L[U])$  if and only if  $0^\dagger$  exists.

*Proof.* ( $\Rightarrow$ ) Let  $x$  be the witness real for  $HP(L[U])$ . Pick  $\lambda > 2^\kappa$  and  $X$  such that  $\lambda$  is  $(x, U)$ -admissible,  $X \prec L_\lambda[U][x]$ ,  $|X| = 2^\kappa$ ,  $X$  is closed under  $\omega$ -sequences and the transitive closure of  $U \cap L_\lambda[U]$  is contained in  $X$ . By Fact 3.7, the transitive collapse of  $X$  is of the form  $L_\theta[U][x]$ . Let  $j : L_\theta[U][x] \cong X \prec L_\lambda[U][x]$  be the inverse of the collapsing map and  $\eta = \text{crit}(j)$ . Note that  $\eta > \kappa$ . Since  $\theta$  is  $(x, U)$ -admissible, by  $HP(L[U])$ ,  $\theta$  is an  $L[U]$ -cardinal. Define  $\bar{U} = \{X \subseteq \eta \mid X \in L[U] \text{ and } \eta \in j(X)\}$ . Since  $(\eta^+)^{L[U]} \leq \theta$ ,  $\bar{U} \subseteq L_\theta[U]$ .  $\bar{U}$  is an  $L[U]$ -ultrafilter on  $\eta$ . Since  $L_\theta[U][x]$  is closed under  $\omega$ -sequences,  $\bar{U}$  is countably complete. So we can build a nontrivial embedding from  $L[U]$  to  $L[U]$  with critical point greater than  $\kappa$ . By Fact 2.26,  $0^\dagger$  exists.

( $\Leftarrow$ ) Suppose  $0^\dagger$  exists and  $\alpha$  is  $0^\dagger$ -admissible. We show that  $\alpha$  is an  $L[U]$ -cardinal. By Fact 3.6, let  $\langle X, Y \rangle$  be the double class of indiscernibles for  $L[U]$ . If  $\alpha \leq \kappa$ , then by Fact 3.8,  $\alpha \in X$ . If  $\alpha > \kappa$ , then by Fact 3.8,  $\alpha \in Y$ . Trivially, elements of  $X$  and  $Y$  are  $L[U]$ -cardinals.  $\square$

**Fact 3.10.** ([13], [16]) Suppose there is no inner model with one measurable cardinal and let  $K$  be the corresponding core model. Then,  $K$  has the rigidity property.

**Corollary 3.11.** (1) Suppose  $0^\#$  exists. Then  $HP(L[0^\#])$  if and only if  $(0^\#)^\#$  exists.  
 (2) Suppose there is no inner model with one measurable cardinal and that  $K$  is the corresponding core model. Then  $HP(K)$  does not hold.

*Proof.* (1) Follows from the proof of “ $HP(L) \Leftrightarrow 0^\#$  exists”. Note that if  $\alpha$  is  $(0^\#)^\#$ -admissible and  $I$  is the class of Silver indiscernibles for  $L[0^\#]$ , then  $I$  is unbounded in  $\alpha$  and hence  $\alpha \in I$ .

(2) Note that  $K = L[\mathcal{M}]$  where  $\mathcal{M}$  is a class of mice. Suppose  $HP(K)$  holds and  $x$  is the witness real for  $HP(K)$ . Pick  $\theta > \omega_2$  and  $X$  such that  $\theta$  is  $(\mathcal{M}, x)$ -admissible,  $X \prec J_\theta[\mathcal{M}, x]$ ,  $\omega_2 \in X$ ,  $|X| = \omega_1$  and  $X$  is closed under  $\omega$ -sequences. Since  $K \models GCH$ , such an  $X$  exists. By the condensation theorem for  $K$ , let  $j : J_{\theta'}[\mathcal{M} \upharpoonright \theta', x] \cong X \prec J_\theta[\mathcal{M}, x]$  be the inverse of the collapsing map. Let

<sup>15</sup>I would like to thank W.Hugh Woodin and Sy Friedman for pointing out this fact to me. The proof of this fact is essentially similar as the proof of the following standard fact: if  $0^\#$  exists,  $I$  is the class of Silver indiscernibles and  $\alpha$  is  $0^\#$ -admissible, then  $I$  is unbounded in  $\alpha$  (see [5, Theorem 4.3]).

$\lambda = \text{crit}(j)$  and  $U = \{X \subseteq \lambda \mid X \in K \text{ and } \lambda \in j(X)\}$ . Note that  $\theta'$  is a  $K$ -cardinal and  $U$  is a countably complete  $K$ -ultrafilter on  $\lambda$ . So there is a nontrivial elementary embedding from  $K$  to  $K$  which contradicts Fact 3.10.  $\square$

From proof of Corollary 3.11(2), if  $M$  is an  $L$ -like inner model,  $M$  has the rigidity property and some proper form of condensation, and  $M \models CH$ , then  $HP(M)$  does not hold.

**Fact 3.12.** ([16])  $(AD^{L(R)})$   $\text{HOD}^{L(R)} = L(P)$  for some  $P \subseteq \Theta$  where  $\Theta = \sup\{\alpha \mid \exists f \in L(R)(f : R \rightarrow \alpha \text{ is surjective})\}$ .

It is an open question whether there exists a nontrivial elementary embedding from  $\text{HOD}$  to  $\text{HOD}$ .<sup>16</sup> However, the following fact shows that the answer to this question is negative for embeddings which are definable in  $V$  from parameters.

**Fact 3.13.** ([9, Theorem 35]) Do not assume  $AC$ . There is no nontrivial elementary embedding from  $\text{HOD}$  to  $\text{HOD}$  that is definable in  $V$  from parameters.

**Theorem 3.14.**  $(ZF + AD^{L(R)})$   $HP(\text{HOD})$  does not hold.

*Proof.* By Fact 3.12, under  $ZF + AD^{L(R)}$ ,  $\text{HOD} = L(P)$  for some  $P \subseteq \Theta$ . Suppose  $HP(\text{HOD})$  holds. Then since  $L(P) \models CH$ , by a similar proof as in Corollary 3.11(2) we can show that there exists a nontrivial elementary embedding  $j : L(P) \rightarrow L(P)$ . Note that  $j$  is definable in  $V$  from parameters. i.e. there is a formula  $\varphi$  and parameter  $\vec{a}$  such that  $j(x) = y$  if and only if  $\varphi(x, y, \vec{a})$ . This contradicts Fact 3.13.  $\square$

#### 4. RELATIONSHIP BETWEEN $HP(L)$ AND THE STRONG REFLECTING PROPERTY FOR $L$ -CARDINALS

In this section, we discuss the relationship between the strong reflecting property for  $L$ -cardinals and Harrington's Principle  $HP(L)$ .

**Theorem 4.1.** (Set forcing)  $SRP^L(\omega_1)$  implies  $Con(Z_2 + HP(L))$ .

*Proof.* Suppose  $SRP^L(\omega_1)$  holds and we want to build a model of  $Z_2 + HP(L)$ . By Proposition 2.8,  $\omega_1$  is limit cardinal in  $L$ . i.e.  $\{\alpha < \omega_1 \mid \alpha \text{ is an } L\text{-cardinal}\}$  is a club. Let  $C = \{\omega \leq \alpha < \omega_1 \mid \alpha \text{ is an } L\text{-cardinal and } L_\alpha \prec L_{\omega_1}\}$ . Note that  $C$  is a club. Let

$$D = \{\gamma < \omega_1 \mid (L_\gamma[C], C \cap \gamma) \prec (L_{\omega_1}[C], C)\}.$$

Note that  $D \subseteq C$ . Define  $F : \omega^\omega \rightarrow \omega^\omega$  as follows: if  $y \subseteq \omega$  codes  $\gamma$ , then  $F(y)$  is a real which codes  $(\beta, C \cap \beta)$  where  $\beta$  is the least element of  $D$  such that  $\beta > \gamma$  (since  $D$  is a club in  $\omega_1$ , such a  $\beta$  exists); if  $y$  does not code an ordinal, let  $F(y) = \emptyset$ .

Let  $\langle \delta_\alpha \mid \alpha < \omega_1 \rangle$  be a pairwise almost disjoint set of reals such that  $\delta_\alpha$  is the  $<_{L[C]}$ -least real which is almost disjoint from any member of  $\{\delta_\beta \mid \beta < \alpha\}$  and  $\langle \delta_\nu \mid \nu < \omega \rangle \in L_\alpha$  for every admissible ordinal  $\alpha < \omega_1$ .

Let  $\langle x_\alpha \mid \alpha < \omega_1 \rangle$  be the enumeration of  $\mathcal{P}(\omega)$  in  $L[C]$  in the order of construction. Let  $Z_F \subseteq \omega_1$  be defined as:

$$Z_F = \{\alpha \cdot \omega + i \mid \alpha < \omega_1 \wedge i \in F(x_\alpha)\}.$$

<sup>16</sup>The answer to this question is negative if  $V = \text{HOD}$ . [9, Theorem 21] provides a very easy proof of the Kunen inconsistency in the case  $V = \text{HOD}$ .

Now we do almost disjoint forcing to code  $Z_F$  via  $\langle \delta_\alpha \mid \alpha < \omega_1 \rangle$ . Then we get a real  $x$  such that  $\alpha \in Z_F \Leftrightarrow |x \cap \delta_\alpha| < \omega$ . The forcing is *c.c.c* and hence preserves all cardinals.

Now we work in  $L[x]$ . Take the least  $\theta$  such that  $L_\theta[x] \models Z_2$ . We will show that  $L_\theta[x] \models HYP(L)$ . By absoluteness, it suffices to show that if  $\alpha < \theta$  is  $x$ -admissible, then  $\alpha$  is an  $L$ -cardinal. Fix some  $x$ -admissible  $\alpha < \theta$  and let

$$\gamma_0 = \sup(\alpha \cap D).$$

If  $\alpha \cap D = \emptyset$ , let  $\gamma_0 = 0$ . Note that if  $\gamma_0 > 0$ , then  $\gamma_0 \in D$ . We assume that  $\gamma_0 < \alpha$  and try to get a contradiction. Let  $\alpha_0$  be the least admissible ordinal such that  $\alpha_0 > \gamma_0$ . Since  $\alpha$  is admissible,  $\alpha_0 \leq \alpha$ .

**Claim 4.2.**  $C \cap \alpha_0 = C \cap (\gamma_0 + 1)$ .

*Proof.* We show that  $C \cap \alpha_0 \subseteq C \cap (\gamma_0 + 1)$ . Suppose  $\gamma \in C \cap \alpha_0$  and  $\gamma > \gamma_0$ . Since  $\gamma \in C$ ,  $L_\gamma \prec L_{\omega_1}$ . Since  $\alpha_0$  is definable from  $\gamma_0$ , it follows that  $\alpha_0$  is definable in  $L_\gamma$ . So  $\alpha_0 \leq \gamma$ . Contradiction.  $\square$

By Claim 4.2,  $L_{\alpha_0}[C] = L_{\alpha_0}[C \cap \gamma_0]$ . We need the following lemma to get that  $L_{\gamma_0}[C \cap \gamma_0][x] = L_{\gamma_0}[x]$  in Claim 4.5.

**Lemma 4.3.**  $C \cap \gamma_0 \in L_{\gamma_0+1}[x]$ .

*Proof.* We prove by induction that for any  $\gamma \in D \cap \theta$ ,  $C \cap \gamma \in L_{\gamma+1}[x]$ . Fix  $\gamma \in D \cap \theta$ . Suppose for any  $\gamma' \in D \cap \gamma$ ,  $C \cap \gamma' \in L_{\gamma'+1}[x]$ . We show that  $C \cap \gamma \in L_{\gamma+1}[x]$ .

Case 1: There is  $\gamma' \in D$  such that  $\gamma$  is the least element of  $D$  such that  $\gamma > \gamma'$ . Let  $\eta$  be the least admissible ordinal such that  $\eta > \gamma'$ . By a similar argument as in Claim 4.2,  $C \cap \eta = C \cap (\gamma' + 1)$ . From our definitions, for any  $\beta < \eta$  we have: (1)  $\langle x_\xi \mid \xi \in \beta \rangle \in L_\eta[C] = L_\eta[C \cap \gamma']$ ; (2)  $\langle \delta_\xi \mid \xi \in \beta \rangle \in L_\eta[C] = L_\eta[C \cap \gamma']$ ; and (3)  $\langle x_\xi \mid \xi \in \eta \rangle$  enumerates  $\mathcal{P}(\omega) \cap L_\eta[C] = \mathcal{P}(\omega) \cap L_\eta[C \cap \gamma']$ .

Suppose  $y \subseteq \omega$  and  $y \in L_\eta[C \cap \gamma']$ . Then  $y = x_\xi$  for some  $\xi < \eta$ . Note that  $\xi \cdot \omega + i < \eta$  for any  $i < \omega$ . Moreover,  $i \in F(y)$  if and only if  $|x \cap \delta_{\xi \cdot \omega + i}| < \omega$ . So  $F(y) \in L_\eta[C \cap \gamma']$ . Hence we have shown that if  $y \in \mathcal{P}(\omega) \cap L_\eta[C \cap \gamma']$ , then  $F(y) \in L_\eta[C \cap \gamma', x]$ .

**Claim 4.4.**  $L_\eta[C \cap \gamma'] \models \gamma' < \omega_1$ .

*Proof.* Suppose, towards a contradiction, that

$$(4.1) \quad \gamma' = \omega_1^{L_\eta[C \cap \gamma']}.$$

Let  $P$  be the almost disjoint forcing that codes  $Z_F$  via the almost disjoint system  $\langle \delta_\beta \mid \beta < \omega_1 \rangle$ .<sup>17</sup> From our definitions of  $C, F$  and  $\langle x_\alpha \mid \alpha < \omega_1 \rangle$ ,  $P$  is a definable subset of  $L_{\omega_1}[C]$ . Standard argument gives that  $P$  is  $\omega_1$ -c.c. in  $L_{\omega_1}[C]$ .<sup>18</sup> Let  $P^* = P \cap L_{\gamma'}[C]$ . Since  $\gamma' \in D$ ,

$$(4.2) \quad (L_{\gamma'}[C], C \cap \gamma') \prec (L_{\omega_1}[C], C).$$

Suppose  $D^* \subseteq P^*$  is a maximal antichain with  $D^* \in L_{\gamma'}[C]$ . Then by (4.2),  $D^*$  is a maximal antichain in  $P$ . Since  $L_{\omega_1}[C] \models D^*$  is at most countable, by (4.2),  $L_{\gamma'}[C] \models D^*$  is at most countable. So  $P^*$  is  $\omega_1$ -c.c. in  $L_{\gamma'}[C]$ . By (4.1),

$$(4.3) \quad L_\eta[C \cap \gamma'] \cap 2^\omega = L_{\gamma'}[C \cap \gamma'] \cap 2^\omega.$$

<sup>17</sup> $P = [\omega]^{<\omega} \times [Z_F]^{<\omega}$ .  $(p, q) \leq (p', q')$  iff  $p \supseteq p', q \supseteq q'$  and  $\forall \alpha \in q' (p \cap \delta_\alpha \subseteq p')$ .

<sup>18</sup>i.e. If  $D \subseteq P$  is a maximal antichain with  $D \in L_{\omega_1}[C]$ , then  $L_{\omega_1}[C] \models D$  is at most countable.

Since  $P^*$  is  $\omega_1$ -c.c. in  $L_{\gamma'}[C]$ , by (4.3),  $P^*$  is  $\omega_1$ -c.c. in  $L_\eta[C \cap \gamma']$ .

We show that  $x$  is generic over  $L_\eta[C \cap \gamma']$  for  $P^*$ . Let  $Y \subseteq P^*$  be a maximal antichain with  $Y \in L_\eta[C \cap \gamma']$ . Since  $P^*$  is  $\omega_1$ -c.c. in  $L_\eta[C \cap \gamma']$ , by (4.1),  $Y \in L_{\gamma'}[C \cap \gamma']$ . By (4.2),  $Y$  is a maximal antichain in  $P$ . So the filter given by  $x$  meets  $Y$ .

Note that  $\gamma' = \omega_1^{L_\eta[C \cap \gamma']} = \omega_1^{L_\eta[C \cap \gamma'][x]}$ . Since  $\gamma' \in D$ , by induction hypothesis  $L_{\gamma'}[C \cap \gamma', x] = L_{\gamma'}[x]$ . So  $L_{\gamma'}[x] \models Z_2$  which contradicts the minimality of  $\theta$ .  $\square$

Take  $y \in L_\eta[C \cap \gamma'] \cap \mathcal{P}(\omega)$  such that  $y$  codes  $\gamma'$ . So  $F(y)$  codes  $(\gamma, C \cap \gamma)$  and  $F(y) \in L_\eta[C \cap \gamma', x]$ . Then  $F(y)$  is definable in  $L_\gamma[C \cap \gamma', x]$ . By induction hypothesis,  $F(y) \in L_{\gamma+1}[x]$ . Since  $F(y)$  codes  $C \cap \gamma$ ,  $C \cap \gamma \in L_{\gamma+1}[x]$ .

Case 2:  $\gamma$  is the least element of  $D$ . Take  $y \in L_\omega[C] \cap \mathcal{P}(\omega)$  such that  $y$  codes 0. Then  $y = x_0$ . Since  $\gamma$  is the least element of  $D$  such that  $\gamma > 0$ ,  $F(y)$  codes  $C \cap \gamma$ . Note that for any  $\beta < \omega$ ,  $\langle \delta_\xi \mid \xi \in \beta \rangle \in L_\omega[C]$  and  $i \in F(y)$  if and only if  $|x \cap \delta_i|$  is finite. So  $F(y)$  is definable in  $L_\omega[x, C]$ . Since  $C \cap \omega = \emptyset$ ,  $F(y) \in L_{\gamma+1}[x]$ . Since  $F(y)$  codes  $C \cap \gamma$ ,  $C \cap \gamma \in L_{\gamma+1}[x]$ .

Case 3:  $\gamma$  is a limit point of  $D$ . Then a standard argument gives that  $C \cap \gamma \in L_{\gamma+1}[x]$  by induction hypothesis.

Since  $\gamma_0 \in D \cap \theta$ , we have  $C \cap \gamma_0 \in L_{\gamma_0+1}[x]$ .  $\square$

**Claim 4.5.**  $\gamma_0$  is countable in  $L_{\alpha_0}[C \cap \gamma_0]$ .

*Proof.* The proof is essentially the same as Claim 4.4 (replace  $\eta$  by  $\alpha_0$  and  $\gamma'$  by  $\gamma_0$ ). Suppose, towards a contradiction, that  $\gamma_0 = \omega_1^{L_{\alpha_0}[C \cap \gamma_0]}$ . By the similar argument as Claim 4.4, we can show that  $x$  is generic over  $L_{\alpha_0}[C \cap \gamma_0]$  for  $P^* = P \cap L_{\gamma_0}[C]$ .<sup>19</sup> Since  $\gamma_0 = \omega_1^{L_{\alpha_0}[C \cap \gamma_0]} = \omega_1^{L_{\alpha_0}[C \cap \gamma_0][x]}$  and by Lemma 4.3,  $L_{\gamma_0}[C \cap \gamma_0][x] = L_{\gamma_0}[x]$ , we have  $L_{\gamma_0}[x] \models Z_2$  which contradicts the minimality of  $\theta$ .  $\square$

From our definitions, we have:

$$(4.4) \quad \text{For } \eta < \alpha_0, \langle \delta_\beta : \beta < \eta \rangle \in L_{\alpha_0}[C] = L_{\alpha_0}[C \cap \gamma_0];$$

$$(4.5) \quad \langle x_\alpha \mid \alpha < \alpha_0 \rangle \text{ enumerates } \mathcal{P}(\omega) \cap L_{\alpha_0}[C] = \mathcal{P}(\omega) \cap L_{\alpha_0}[C \cap \gamma_0].$$

**Claim 4.6.** If  $y \in \mathcal{P}(\omega) \cap L_{\alpha_0}[C \cap \gamma_0]$ , then  $F(y) \in L_{\alpha_0}[x]$ .

*Proof.* Suppose  $y \in \mathcal{P}(\omega) \cap L_{\alpha_0}[C \cap \gamma_0]$ . By (4.5),  $y = x_\xi$  for some  $\xi < \alpha_0$ . Note that for  $\xi < \alpha_0$ ,  $\xi \cdot \omega + i < \alpha_0$  for any  $i \in \omega$ . By the definition of  $Z_F$ ,  $i \in F(y) \Leftrightarrow \xi \cdot \omega + i \in Z_F \Leftrightarrow |x \cap \delta_{\xi \cdot \omega + i}| < \omega$ . By (4.4),  $F(y) \in L_{\alpha_0}[C \cap \gamma_0][x]$ . Since  $C \cap \gamma_0 \in L_{\gamma_0+1}[x]$  by Lemma 4.3, we have  $L_{\alpha_0}[C \cap \gamma_0][x] = L_{\alpha_0}[x]$ . So  $F(y) \in L_{\alpha_0}[x]$ .  $\square$

By Claim 4.5, there exists a real  $y \in L_{\alpha_0}[C \cap \gamma_0]$  such that  $y$  codes  $\gamma_0$ . Note that  $F(y)$  codes  $\gamma_1$  where  $\gamma_1$  is the least element of  $C$  such that  $\gamma_1 > \gamma_0$  and  $(L_{\gamma_1}[C], C \cap \gamma_1) \prec (L_{\omega_1}[C], C)$ . Since  $F(y)$  codes  $\gamma_1$  and  $F(y) \in L_{\alpha_0}[x]$ ,  $\gamma_1 < \alpha_0$ . Since  $\gamma_1 < \alpha$  and  $(L_{\gamma_1}[C], C \cap \gamma_1) \prec (L_{\omega_1}[C], C)$ , by the definition of  $\gamma_0$ , we have that  $\gamma_1 \leq \gamma_0$ . Contradiction.

So the assumption  $\gamma_0 < \alpha$  is false. Then  $\gamma_0 = \alpha$ . So  $\alpha \in C$  and hence  $\alpha$  is an  $L$ -cardinal. We have shown that  $L_\theta[x] \models Z_2 + HP(L)$ .  $\square$

**Theorem 4.7.** ([2, Theorem 3.1, 3.2]) *(Class forcing)  $Z_2 + HP(L)$  is equiconsistent with ZFC and  $Z_3 + HP(L)$  is equiconsistent with ZFC + there exists a remarkable cardinal.*

<sup>19</sup> $P$  is the almost disjoint forcing that codes  $Z_F$  via  $\langle \delta_\beta \mid \beta < \omega_1 \rangle$ .

**Corollary 4.8.** (a) For  $n \geq 3$ ,  $SRP^L(\omega_n)$  is equivalent to  $HP(L)$ .  
 (b) (Set forcing)  $SRP^L(\omega_2)$  is strictly stronger than  $Z_3 + HP(L)$ .  
 (c) (Set forcing)  $SRP^L(\omega_1)$  is strictly stronger than  $Z_2 + HP(L)$ .

*Proof.* (a) follows from Theorem 2.23 and Theorem 3.3. (b) follows from Theorem 2.17 and Theorem 4.7. (c) follows from Theorem 4.1, Theorem 4.7 and Proposition 2.8.  $\square$

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